Sparse Sensing in Colocated MIMO Radar: A Matrix Completion Approach

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Outline

1. Motivation
2. MIMO Radars vs Phased Arrays
3. Compressive Sensing Based MIMO Radars (CS-MIMO)
4. MIMO Radars based on Matrix Completion (MC-MIMO)
Motivation

- There is increasing interest in networked radars that are inexpensive and enable reliable surveillance. Unfortunately, these requirements are competing in nature.
- In a networked radar the processing can be done at a fusion center, which collects the measurements of all receive antennas. Reliable surveillance requires collection, communication and fusion of vast amounts of data from various antennas, which is a bandwidth and power intensive task.
- The communication with the fusion center could occur via a wireless link (radar on a wireless sensor network).
- MIMO radars have received considerable recent attention as they can achieve superior resolution.
- The talk presents new results on networked MIMO radars that rely on advanced signal processing, and in particular, sparse sensing and matrix completion, in order to achieve an optimal tradeoff between reliability and cost (bandwidth, power).
- These techniques will enable the radar to meet the same operational objectives with traditional MIMO radars while involving significantly fewer samples, be robust, and operate on mobile platforms.
A phased array radar

- is composed of many closely spaced antennas
- all antennas transmit the same waveform
- is capable of cohering and steering the transmit energy
Multiple input multiple output (MIMO) radar

- employs colocated TX/RX antennas or widely separated TX/RX antennas;
- uses multiple waveforms:
  - Independent waveforms $\Rightarrow$ omnidirectional beampattern
  - Correlated waveforms $\Rightarrow$ desired beampattern

Colocated MIMO Radar
[Xu et. al. 2006, Li et. al. 2007, Li et. al. 2008]

Widely separated MIMO Radar
[Fisher et. al. 2004, Lehmann et. al. 2006, Haimovich et. al. 2008]
CS-MIMO radar - Signal model (1)

[Yu, Petropulu and Poor, IEEE JSTSP, 2010]

- $M_t$ transmit and $N_r$ receive colocated nodes are employed;
- there are $K$ point targets in the far field and multiple jammers;
- the $k$-th target is at azimuth angle $\theta_k$, moving with constant radial speed $v_k$ and of range $d_k(t)$;
- the distance between the $i$th TX/RX antenna and the $k$th target
  \[ d_{ik}^{t/r}(t) \approx d_k(0) - v_k t - \eta_i^{t/r}(\theta_k), \]
  where \( \eta_i^{t/r}(\theta_k) = r_i^{t/r} \cos(\theta_k - \alpha_i^{t/r}) \);
- the transmit nodes transmit pulses of pulse repetition interval (PRI) $T$. 

\[ \theta_k - \alpha_i^{t/r} = \gamma_i^{t/r} \]
The signal reflected by the $k$-th target equals

$$y_k(t) = \beta_k \sum_{i=1}^{M_t} x_i(t - d_{ik}^t(t)/c) e^{j2\pi f(t - \frac{d_{ik}^t(t)}{c})}$$

The signal at the $l$-th receive antenna equals

$$z_l(t) = \sum_{k=1}^{K} z_l(t - \frac{d_{lk}^r(t)}{c}) + \epsilon_l(t)$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{M_t} \beta_k x_i(t - \frac{d_{ik}^t(t) + d_{lk}^r(t)}{c}) e^{j2\pi f(t - \frac{d_{ik}^t(t) + d_{lk}^r(t)}{c})} + \epsilon_l(t)$$

$\epsilon_l(t)$: i.i.d. Gaussian noise with zero mean and variance $\sigma^2$. 

CS-MIMO Radars - Signal Model (2)
The transmit waveforms, $x_i(t), i = 1, \ldots, M_t$, are narrowband.

The targets are moving slowly, i.e., it holds $\frac{2vfT_P}{c} \ll 1$.

Received baseband signal at the $l$-th antenna:

$$z_l(t) \approx \sum_{k=1}^K \sum_{i=1}^{M_t} \beta_k x_i(t) e^{j2\pi f\left(t - \frac{d_{lk}(t) + d_{ik}(t)}{c}\right)} + \epsilon_l(t)$$

$$= \sum_{k=1}^K \beta_k e^{-j\frac{2\pi}{\lambda} 2d_k(0)} e^{j\frac{2\pi}{\lambda} \eta_l^t(\theta_k)} e^{j2\pi f_k t} x^T(t) v(\theta_k) + \epsilon_l(t)$$

- $\lambda$: the transmitted signal wavelength
- $f_k = 2v_k f / c$: the Doppler shift caused by the $k$-th target
- $v(\theta_k) = [e^{j\frac{2\pi}{\lambda} \eta_1^t(\theta_k)}, \ldots, e^{j\frac{2\pi}{\lambda} \eta_{Mt}^t(\theta_k)}]^T$
- $x(t) = [x_1(t), \ldots, x_{M_t}(t)]^T$
On letting $L$ denote the number of snapshots and $T_s$ the sampling period, the received samples collected during the $m$-th pulse are given by

$$z_{lm} = \begin{bmatrix} z_l((m-1)T + 0T_s) \\ \vdots \\ z_l((m-1)T + (L-1)T_s) \end{bmatrix} = \sum_{k=1}^{K} \gamma_k e^{j\frac{2\pi}{\lambda} \eta_l(\theta_k)} e^{j2\pi f_k (m-1)T} D(f_k) X v(\theta_k) + e_{lm}$$

where

$$\gamma_k = \beta_k e^{-j\frac{2\pi}{\lambda} 2d_k(0)}, \quad D(f_k) = \text{diag}\{[e^{j2\pi f_k 0 T_s}, \ldots, e^{j2\pi f_k (L-1)T_s}]\}$$

$$e_{lm} = [\epsilon_l((m-1)T + 0T_s), \ldots, \epsilon_l((m-1)T + (L-1)T_s)]^T$$

$$X = [x(0T_s), \ldots, x((L-1)T_s)]^T \quad (L \times M_t).$$
Let us discretize the angle-Doppler plane on a fine grid as:

\[ \mathbf{a} = [(a_1, b_1), \ldots, (a_N, b_N)] \] (5)
Rewrite as
\[ z_{lm} = \sum_{n=1}^{N} s_n e^{j \frac{2\pi}{\lambda} \eta_i r (a_n)} e^{j 2\pi b_n (m-1) T} \mathbf{D}(b_n) \mathbf{Xv}(a_n) + e_{lm} \] (6)

where
\[ s_n = \begin{cases} \gamma_k, & \text{if the } k\text{-th target is at } (a_n, b_n) \\ 0, & \text{otherwise} \end{cases} \]

In matrix form we have
\[ z_{lm} = \mathbf{\Psi}_{lm} \mathbf{s} + e_{lm} \] (7)

where
\[ \mathbf{s} = [s_1, \ldots, s_N]^T \]
\[ \mathbf{\Psi}_{lm} = [e^{j \frac{2\pi}{\lambda} \eta_i r (a_1)} e^{j 2\pi b_1 (m-1) T} \mathbf{D}(b_1) \mathbf{Xv}(a_1), \ldots, e^{j \frac{2\pi}{\lambda} \eta_i r (a_N)} e^{j 2\pi b_N (m-1) T} \mathbf{D}(b_N) \mathbf{Xv}(a_N)] \]
Rewrite as

\[ z_{lm} = \sum_{n=1}^{N} s_n e^{j \frac{2\pi}{\lambda} \eta^r_i(a_n)} e^{j 2\pi b_n (m-1) T} D(b_n) X_v(a_n) + e_{lm} \]  

(6)

where

\[ s_n = \begin{cases} 
\gamma_k, & \text{if the } k\text{-th target is at } (a_n, b_n) \\
0, & \text{otherwise}
\end{cases} \]

In matrix form we have

\[ z_{lm} = \Psi_{lm} s + e_{lm} \]  

(7)

where

\[ s = [s_1, \ldots, s_N]^T \]

\[ \Psi_{lm} = [e^{j \frac{2\pi}{\lambda} \eta^r_i(a_1)} e^{j 2\pi b_1 (m-1) T} D(b_1) X_v(a_1), \ldots, e^{j \frac{2\pi}{\lambda} \eta^r_i(a_N)} e^{j 2\pi b_N (m-1) T} D(b_N) X_v(a_N)] \]

The number of targets is small

\[ \Rightarrow \text{the positions of targets are sparse in the angle-Doppler plane} \]

\[ \Rightarrow s \text{ is a sparse vector.} \]
The $l$th node obtains measurements as:

$$r_{lm} = \Phi_{lm}z_{lm} = \Phi_{lm}\Psi_{lm}s + \tilde{e}_{lm}, \quad (8)$$

$N_r$ receive nodes forward the compressed measurements collected during $N_p$ pulses to a fusion center.

The fusion center combines all the received data as

$$r = [r_{11}^T, \ldots, r_{1N_p}^T, \ldots, r_{N_l1}^T, \ldots, r_{N_lN_p}^T]^T = \Theta s + E \quad (9)$$

$s$ is recovered by applying the Dantzig selector to the above convex problem [Candes and Tao 2007].


CS-MIMO Radars can achieve the same resolution as MIMO radars but with significantly fewer samples, or better resolution with the same number of samples.

CS-MIMO Radars need grid discretization.

We next present MC-MIMO Radars, which maintain the advantages of CS-MIMO Radars but do not rely on grid discretization.
Matrix Completion

Recover a rank $r$ matrix $\mathbf{M} \in \mathbb{C}^{N_1 \times N_2}$ based on partial knowledge of its entries. Define $\mathbf{Y} = \mathcal{P}_\Omega(\mathbf{M})$ as

$$[\mathbf{Y}]_{ij} = \begin{cases} [\mathbf{M}]_{ij}, & (i, j) \in \Omega; \\ 0, & \text{otherwise}. \end{cases}$$

where $\Omega$ is the set of indices of observed entries with cardinality $m$.

Under certain conditions, $\mathbf{M}$ can be recovered by solving the following optimization problem:

$$\min \| \mathbf{X} \|_*$$
$$\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M})$$

(10)

Nuclear norm minimization is the tightest convex relaxation of the NP-hard rank minimization problem.

[Candes & Recht, 2009], [Candes & Tao, 2010], [Candes & Plan, 2010]
Definition

(Subspace Coherence) Let \( U \equiv \mathbb{C}^r \subseteq \mathbb{C}^N \) be a subspace spanned by the set of orthonormal vectors \( \{u_i \in \mathbb{C}^{N \times 1}\}_{i \in \mathbb{N}_r^+} \). Also, define the matrix

\[
\mathbf{U} \triangleq [u_1 \ u_2 \ldots u_r] \in \mathbb{C}^{N \times r}
\]

and let \( \mathbf{P}_U \equiv \mathbf{UU}^H \in \mathbb{C}^{N \times N} \) be the orthogonal projection onto \( U \). Then, the coherence of \( U \) with respect to the standard basis \( \{\mathbf{e}_i\}_{i \in \mathbb{N}_N^+} \) is defined as

\[
\mu(U) \triangleq \frac{N}{r} \sup_{i \in \mathbb{N}_N^+} \|\mathbf{P}_U \mathbf{e}_i\|_2^2 \in \left[1, \frac{N}{r}\right].
\]  

(11)

Concerning the recoverability of \( \mathbf{M} \), the following assumptions regarding the subspaces \( U \) and \( V \) are of particular importance [?].

**A0** \( \max \{\mu(U), \mu(V)\} \leq \mu_0 \in \mathbb{R}_{++} \).

**A1** \( \left\| \sum_{i \in \mathbb{N}_r^+} u_i v_i^H \right\|_\infty \leq \mu_1 \sqrt{\frac{r}{N_1 N_2}}, \quad \mu_1 \in \mathbb{R}_{++} \).
(Exact MC) Let $\mathbf{M} \in \mathbb{C}^{N_1 \times N_2}$ be a matrix of bounded rank $r$ obeying the incoherence property with bounded parameter $\mu_0$ and set $N \triangleq \max\{N_1, N_2\}$. Suppose we observe $m$ entries of $\mathbf{M}$ with locations sampled uniformly at random. Then, there exist positive numerical constants $C_1$ and $C_2$ such that if

$$m \geq C_1 \mu_0^4 N r^4 \log^2 N = O\left(N \log^2 N\right) \quad \text{or} \quad (12)$$

$$m \geq C_2 \mu_0^2 N r^2 \log^6 N = O\left(N \log^6 N\right), \quad (13)$$

the minimizer to the program (10) is unique and equal to $\mathbf{M}$ with probability at least $1 - N^{-3}$. 

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Sparse Sensing in Colocated MIMO Radar: A Matrix Completion Approach 
July 8, 2014 17 / 43
An $N_1 \times N_2$ matrix of rank $r$ depends upon $N_1 N_2 - (N_1 - r)(N_2 - r)$ degrees of freedom. When $r$ is small the number of d.f. is smaller than $N_1 N_2$, and this is why subsampling is possible.

In compressed sensing, the number of d.f. corresponds to the sparsity of the signal, (i.e., number of non-zero entries)

It is remarkable that exact recovery occurs as soon as the sample size exceeds the number of d.f. by a couple of logarithmic factors

If the sampling misses one column or one row the one cannot hope to recover the matrix, even if the matrix has rank one.

One needs to sample at random. It is established that one needs $O(N \log N)$ samples for this to happen ($N = \max(N_1, N_2)$).
Matrix Completion

Recover a rank $r$ matrix $M \in \mathbb{C}^{N_1 \times N_2}$ based on partial knowledge of its entries.

Let $Y = \mathcal{P}_\Omega (M + E)$ as

$$[Y]_{ij} = \begin{cases} [M]_{ij} + [E]_{ij}, & (i, j) \in \Omega \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (14)

where $E$ is noise, and $\Omega$ is the set of indices of observed entries with cardinality $m$. Under certain conditions, $M$ can be recovered by solving the following optimization problem:

$$\min \|X\|_* \quad \text{s.t. } \|P_\Omega (X - Y)\|_F \leq \delta.$$  \hspace{1cm} (15)

Assuming that the noise is zero-mean, white, $\delta$ is related to the noise variance, $\sigma^2$, as $\delta^2 = (m + \sqrt{8m})\sigma^2$. 

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When observations are corrupted with white zero-mean Gaussian noise with variance $\sigma^2$, the recovery error is bounded as

$$
\| M - \hat{M} \|_F \leq 4 \sqrt{\frac{1}{p} (2 + p) \min(n_1, n_2) \delta + 2\delta},
$$

(16)

where $p = \frac{m}{N_1 N_2}$ is the fraction of observed entries, and $\delta^2 = (m + \sqrt{8m})\sigma^2$. 
Transmission antennas transmit \textit{narrowband and orthogonal} waveforms, that is,
\[ \frac{1}{T_p} \ll \frac{c}{\lambda}, \tag{17} \]
where \( T_p \in \mathbb{R} \), \( \lambda \in \mathbb{R} \) and \( c \equiv 3 \cdot 10^8 \text{ m/s} \) denotes the waveform duration, the communication wavelength and the speed of light, respectively.

The target reflection coefficients \( \{\beta_i \in \mathbb{C}\}_{i \in \mathbb{N}_K^+} \) (\( K \) is the number of targets in the far field) remain constant during a number of pulses \( Q \).

The delay spread in the received signals is smaller that the temporal support of each waveform \( T_p \).

The Doppler spread of the received signals is much smaller than the bandwidth of the pulse, that is,
\[ \frac{2v_i}{\lambda} \ll \frac{1}{T_p}, \quad \forall i \in \mathbb{N}_K^+ \tag{18} \]
where \( v_i \in \mathbb{R} \) denotes the speed of the respective target.
MIMO Radar with Matric Completion (MC-MIMO)

- orthogonal transmit waveforms,
- \( K \) targets

\[
Y = \begin{bmatrix}
1 & \cdots & M_r - 1 & M_r \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & M_t & \cdots & M_t
\end{bmatrix}
\]
It can be shown that the fully observed version of the data matrix formulated at the fusion center can be expressed as

\[
\mathbf{Y} \triangleq \mathbf{\Delta} + \mathbf{Z} \in \mathbb{C}^{M_r \times M_t},
\]

(19)

where \( \mathbf{Z} \) is an interference/observation noise matrix that may also describe model mismatch due to weak correlations among the transmit waveforms and

\[
\mathbf{\Delta} \triangleq \mathbf{X}_r \mathbf{D} \mathbf{X}_t^T,
\]

(20)

where \( \mathbf{X}_r \in \mathbb{C}^{M_r \times K} \) (respectively for \( \mathbf{X}_t \in \mathbb{C}^{M_t \times K} \)) constitutes an *alternant* matrix defined as

\[
\mathbf{X}_r \triangleq \begin{bmatrix}
\gamma_0^0 & \gamma_1^0 & \ldots & \gamma_{K-1}^0 \\
\gamma_0^1 & \gamma_1^1 & \ldots & \gamma_{K-1}^1 \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{M_r-1}^0 & \gamma_{M_r-1}^1 & \ldots & \gamma_{M_r-1}^{K-1}
\end{bmatrix} \in \mathbb{C}^{M_r \times K},
\]

(21)
...with

\[ \gamma_k^l \triangleq e^{j2\pi r_r^T(l)T(\theta_k)}, \quad (l, k) \in \mathbb{N}_{M_r-1} \times \mathbb{N}_{K-1} \] (22)

\[ r_r(l) \triangleq \frac{1}{\lambda} [x^r_l \ y^r_l]^T \in \mathbb{R}^{2\times1}, \quad l \in \mathbb{N}_{M_r-1} \quad \text{and} \]

\[ T(\theta_k) \triangleq \begin{bmatrix} \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix} \in \mathbb{R}^{2\times1}, \quad k \in \mathbb{N}_{K-1}. \] (24)

- The sets \( \left\{ [x^r_l \ y^r_l]^T \right\}_{l \in \mathbb{N}_{M_r-1}} \) and \( \left\{ \theta_k \right\}_{k \in \mathbb{N}_{K-1}} \) contain the 2-dimensional antenna coordinates of the reception array and the target angles, respectively,
- \( \lambda \in \mathbb{R}_{++} \) denotes the carrier wavelength, and
- \( D \in \mathbb{C}^{K \times K} \) is a non-zero diagonal matrix whose elements depend on the target reflection properties and the speeds.

For the simplest ULA case,

\[ \begin{bmatrix} x^r_l(r(t)) \\ y^r_l(r(t)) \end{bmatrix}^T \equiv [0 \ ld_r(t)]^T, \quad l \in \mathbb{N}_{M_r(t)-1}. \] (25)

and \( X_r \) and \( X_t \) degenerate to Vandermonde matrices.
Sparse Sensing is implemented through the following Random Matched Filter Bank (RMFB) architecture.

- Simple power saving Bernoulli switching with selection probability $p$.
- RMFBs are implemented in receiver, constructing the Bernoulli subsampled version of $\Delta$, $\mathcal{P}(\Delta)$.
- Matrix Completion is applied for the stable recovery of $\Delta$.
- The recovered matrix is fed into standard array processing methods (e.g. MUSIC) for extracting target information.
MC-MIMO Radar: Sampling Scheme I


\[ Y = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \]
Coherence bounds - Simulations results

Scheme I, $K = 2$ targets: (a) the average $\max(\mu(U), \mu(V))$ of $Z^\text{MF}_q$ as function of number of transmit and receive antennas, and for $\Delta\theta = 5^\circ$; (b) the average $\max(\mu(U), \mu(V))$ of $Z^\text{MF}_q$ as function of DOA separation.
DOA resolution - Simulations results

Scheme I: DOA resolution. The parameters are set as $M_r = M_t = 20$, $p_1 = 0.5$ and $\text{SNR} = 10, 25\,\text{dB}$. 
Scheme I, $K = 2$ targets: the relative recovery error for $Z_q^{MF}$ under different values of DOA separation. $M_r = M_t = 40$. 

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Sparse Sensing in Colocated MIMO Radar: A Matrix  
July 8, 2014 29 / 43
Suppose that the Nyquist rate samples of the signals at the RX nodes correspond to sampling times $t_i = iT_s$, $i \in \mathbb{N}_{N-1}$ with $N = T_p/T_s$.

Let the $l$th receive antenna sample at times $\tau_j^l = jT_s$, $j \in \mathcal{J}^l$, where $\mathcal{J}^l$ is the output of a random number generator, containing $L_2$ integers in the interval $[0, N - 1]$.

The $l$th receive antenna forwards the $L_2$ samples to the fusion center.
In this case, the fully observed version of the data matrix is of the form

\[ \Delta \triangleq X_r DX_t^T S, \]  

where \( S \in \mathbb{C}^{M_t \times N} \) contains as its rows the waveform samples of the transmission antennas.

Assuming that \( N \geq M_t \geq K \), \( \Delta \) will be a rank-\( K \) matrix.

As before, matrix completion is applied for the stable recovery of \( \Delta \).

The recovered matrix is fed into standard array processing methods (e.g., MUSIC) for extracting target information.

However, in this case we have to make use of the assumed discrete orthogonality of \( S \),

\[ SS^H \equiv I. \]
MC estimation error and probability of target resolution - Simulations results

- Consider ULAs with $M_t = 20$, $M_r = 40$.
- The SNR is set to $25\, \text{dB}$.

![Graph showing the relationship between reciprocal of SNR and probability of resolution for different matrix completion schemes.](image-url)
Recoverability & Performance Guarantees: 
Scheme I [Kalogerias & Petropulu, 2013, 2014]

Useful results:

**Theorem**

[Wolkowicz 1980] Let $\mathbf{M} \in \mathbb{C}^{N \times N}$ be a matrix with real eigenvalues. Define

$$
\tau \triangleq \frac{\text{tr} (\mathbf{M})}{N} \quad \text{and} \quad s^2 \triangleq \frac{\text{tr} (\mathbf{M}^2)}{N} - \tau^2.
$$

(28)

Then, it is true that

$$
\tau - s \sqrt{N-1} \leq \lambda_{\text{min}} (\mathbf{M}) \leq \tau - \frac{s}{\sqrt{N-1}} \quad \text{and} \quad \tau + \frac{s}{\sqrt{N-1}} \leq \lambda_{\text{max}} (\mathbf{M}) \leq \tau + s \sqrt{N-1}.
$$

(29) (30)

Further, equality holds on the left (right) of (29) if and only if equality holds on the left (right) of (30) if and only if the $N-1$ largest (smallest) eigenvalues are equal.
We can show that if \( \Delta = X_rD X_t^T \in \mathbb{C}^{M_r \times M_t} \),

\[ \mu(U) \leq \frac{M_r}{\lambda_{\min}(X_r^H X_r)} \quad \text{and} \quad \mu(V) \leq \frac{M_t}{\lambda_{\min}(X_t^H X_t)} . \]

- For a ULA, the elements of \( X_t^H X_t \) (respectively for \( X_r^H X_r \)) are of the form

\[ \delta_{i,j} \triangleq \sum_{m=0}^{M_t-1} e^{j2\pi m(\alpha_t^i - \alpha_t^j)}, \quad \forall (i,j) \in \mathbb{N}_{K-1} \times \mathbb{N}_{K-1}. \quad (31) \]

- The trace of \( X_t^H X_t \) is \( M_t K \).
$\text{tr} \left( (X_t^H X_t)^2 \right) = \sum_{k_1=0}^{K-1} M_t^2 + \sum_{k_2=0}^{K-1} \sum_{k_1 \neq k_2} \frac{\sin^2 \left( \pi M_t \left( \alpha_{k_1}^t - \alpha_{k_2}^t \right) \right)}{\sin^2 \left( \pi \left( \alpha_{k_1}^t - \alpha_{k_2}^t \right) \right)}$ 

$\equiv \sum_{k_1=0}^{K-1} M_t^2 + \sum_{k_2=0}^{K-1} \phi^2_{M_t} \left( \alpha_{k_1}^t - \alpha_{k_2}^t \right) \leq \sum_{k_1=0}^{K-1} \left( M_t^2 + (K-1) \sup_{x \in [\xi_t, \frac{1}{2}]} \phi^2_{M_t} (x) \right)$ 

$\triangleq KM_t^2 + K (K - 1) \beta \xi_t (M_t)$.

$\alpha_k^r \triangleq \frac{d_r \sin (\theta_k)}{\lambda}$ 

$\xi_t : \text{smallest } \alpha_i^t - \alpha_j^t \text{ folded in } [0, \frac{1}{2}]$
Theorem (Brief Version) (Coherence for ULAs) Consider a Uniform Linear Array (ULA) transmission - reception pair and assume that the set of target angles $\{\theta_k\}_{k \in \mathbb{N}_{K-1}}$ consists of almost surely distinct members. Then, for any fixed $M_t$ and $M_r$, as long as

$$K \leq \min_{i \in \{t,r\}} \left\{ \frac{M_i}{\sqrt{\beta \xi_i (M_i)}} \right\},$$

(32)

the associated matrix $\Delta$ obeys the assumptions $A0$ and $A1$ with

$$\mu_0 \triangleq \max_{i \in \{t,r\}} \left\{ \frac{M_i}{M_i - (K - 1) \sqrt{\beta \xi_i (M_i)}} \right\} \quad \text{and} \quad \mu_1 \triangleq \mu_0 \sqrt{K}.$$

In the above $\sqrt{\beta \xi_i (M_i)}$ denotes a constant dependent on $M_i$. $\xi_i, i \in \{t, r\}$, mostly depends on the pairwise differences $|\sin (\theta_i) - \sin (\theta_j)|$, $(i, j) \in \mathbb{N}_{K-1} \times \mathbb{N}_{K-1}, i \neq j$. 

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Further, if \( \xi \triangleq \min \{ \xi_r, \xi_t \} \neq 0 \), then, for any fixed \( K \), as long as

\[
\min_{i \in \{t, r\}} M_i \geq K \sqrt{\beta \xi} \equiv \frac{K}{\sin(\pi \xi)} = \mathcal{O}(K),
\]

both \( \beta \xi_t (M_t) \) and \( \beta \xi_r (M_r) \) can be replaced by the constant \( \beta \xi \) (that is, independent of both \( M_t \) and \( M_r \)). Additionally, in the limit with respect to \( M_t \) and \( M_r \), we have

\[
\mu(V) \equiv \mu(U) \equiv 1,
\]

that is, the coherence of \( \Delta \) is asymptotically optimal.

In other words, under very reasonable assumptions, the coherence of \( \Delta \) is both asymptotically and approximately optimal with respect to the number of transmission and reception antennas.
In colocated MIMO Radar systems, due to the need for unambiguous angle estimation (target detection), it is very common to assume that $\theta_i \in [-\pi/2, \pi/2], \forall i \in \mathbb{N}_{K-1}$.

Also, especially for the case where ULAs are employed for transmission and reception, another common assumption is to choose $d_r \equiv d_t = \lambda/2$.

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**Lemma**

(ULA Pairs Coherence Control) *Consider the hypotheses and definitions of Theorem 2 and set* $d_r \equiv d_t = \lambda/2$. *If additionally*

$$\xi = 1 - \cos\left(\frac{\eta}{2}\right) \in \left(0, \frac{2 - \sqrt{2}}{2}\right)$$

*and the asymptotics of Theorem 2 always hold true. In particular, the higher the value of* $\eta$, *the higher the value of* $\xi$ *and the lower the coherence of* $\Delta$. 
Theorem

(Coherence for Arbitrary 2D Arrays) Let the abstract sets \(\mathcal{I}\) and \(\mathcal{R}\) contain all the essential information regarding the transmitter and receiver array topologies, and assume that the target angles are distinct. Then, for any \(M_t\) and \(M_r\), as long as

\[ K \leq \min_{i \in \{t, r\}} \left\{ \frac{M_i}{\sqrt{\beta_i}} \right\}, \tag{38} \]

the associated matrix \(\Delta\) obeys the assumptions \(A0\) and \(A1\) with

\[ \mu_0 \triangleq \max_{i \in \{t, r\}} \left\{ \frac{M_i}{M_i - (K - 1) \sqrt{\beta_i}} \right\} \quad \text{and} \quad \mu_1 \triangleq \mu_0 \sqrt{K}. \tag{39} \]

with probability 1 and where

\[ \beta_{t(r)} \triangleq \sup_{(x, y) \in \mathcal{A}} \left| \varphi_{t(r)}(x, y|\mathcal{I}(\mathcal{R})) \right|^2 \in \left[0, M_{t(r)}^2\right], \tag{40} \]

with

\[ \varphi_{t(r)}(x, y|\mathcal{I}(\mathcal{R})) \triangleq \sum_{m=0}^{M_{t(r)}-1} \exp \left(j2\pi r_{t(r)}^T(m)(\mathcal{T}(x) - \mathcal{T}(y))\right), \tag{41} \]
Consider a MIMO Radar system equipped with identical Uniform Circular Arrays (UCAs) with $M_r = M_t = 20 \triangleq M$ antennas.

Their positions in the 2-dimensional plane are defined as

$$
\begin{bmatrix}
    x_i^{t(r)} \\
    y_i^{t(r)}
\end{bmatrix} \triangleq R \begin{bmatrix}
    \cos \left(\frac{2\pi (l - 1)}{M}\right) \\
    \sin \left(\frac{2\pi (l - 1)}{M}\right)
\end{bmatrix},
$$

$I \in \mathbb{N}_{M-1}$, where $R = 0.5 \, m$.

The wavelength utilized for the communication is chosen as $\lambda = 0.5 \, m$. 
The function $|\varphi_{t(r)}(x, y | \mathcal{R})|^2$ of Example 1 with respect to $(x, y) \in [-\pi, \pi]^2$ for the case of (a) a symmetric UCA pair and (b) a symmetric ULA pair.
Conclusions

- We have investigated the problem of reducing the volume of data typically required for accurate target detection and estimation in colocated MIMO radars.
- We have presented two sparse sensing schemes (Schemes I & II) for information acquisition, leading to the natural formulation of a low rank matrix completion problem, which can be efficiently solved using convex optimization.
- Numerical simulations have justified the effectiveness of our approach.
- Specifically for Scheme I, we have presented theoretical results, guaranteeing near optimal performance of the respective matrix completion problem, for the case where ULAs are employed for transmission and reception.
Relevant Publications


